J Glob Optim (2008) 40:161–168
DOI 10.1007/s10898-007-9188-3

Second-order conditions in stability analysis for state constrained optimal control

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Received: 30 May 2007 / Accepted: 31 May 2007 / Published online: 6 July 2007

Abstract The paper presents an outline of the stability results, for state-constrained optimal control problems, recently obtained in Malanowski (Appl. Math. Optim. 55, 255–271, 2007), Malanowski (Optimization, to be published), Malanowski (SIAM J. Optim., to be published). The pricipal novelty of the results is a weakening of the second-order sufficient optimality conditions, under which the solutions and the Lagrange multipliers are locally Lipschitz continuous functions of the parameter. The conditions are weakened by taking into account strongly active state constraints.

Keywords Parametric optimal control · Nonlinear ODEs · State constraints · Second order sufficient conditions · Lipschitz stability of the solutions

1 Introduction

In stability analysis for optimal control problems, conditions are investigated, under which the solutions and the associated Lagrange multipliers are locally Lipschitz continuous functions of the parameters. It is known that these conditions consist of constraint qualifications and second-order sufficient optimality conditions, which should be satisfied at the reference point. For control constrained problems a complete characterization of the Lipschitz stability was derived (see [2,6]).The situation is different for problems with first-order state constraints, where a strong second-order sufficient optimality conditions were used [1,5]. It is connected with the fact that, under small perturbations, the behavior of control and state constraints are very different from each other. Namely, control constraints, strongly active at the reference solution (i.e., such that the values of the corresponding multipliers are greater than a positive constant), remain active under small perturbations, whereas the example presented in [8] shows that the structure of strongly active state constraints can be changed by arbitrary small perturbations. This was the reason why in papers [1,5], devoted to stability analysis

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for state-constrained problems, a strong second-order condition was used, where active state constraints were not taken into account. Thus, there was a substantial gap between sufficient and necessary conditions for the local Lipschitz stability.

In the series of the recent papers [8–10], it was shown that, in spite of instability of the active state constraints, the weak second-order conditions remain stable under small perturbations. Thus, the second-order conditions, weakened by taking into account the strongly active state constraints, can be used in stability analysis, in a similar way as for control-constrained problems.

The present paper gives an outline of the results obtained in [8,10]. The emphasis is on the basic mechanisms, rather than on technicalities. In Sect. 2, the considered optimal control problems are formulated and the needed assumptions are introduced. Some basic results concerning state-constrained optimal control problems are recalled. Section 3 is devoted to stability analysis. The basic auxiliary lemmas and the principal stability results are formulated. For the proofs, the reader is refered to [10].

2 Parametric optimal control

In this section, our model parametric optimal control problem is formulated and basic assumptions are introduced. Let $Z = L^2(0, 1; I\!R)$ and $H = W^{2,\infty}$ be the spaces of parameters and feasible parameters, respectively. Moreover, denote by

$$X^p = W_0^{1,p}(0, 1; I\!R^n) \times L^p(0, 1; I\!R^m), \quad p \in [1, \infty]$$

the spaces of arguments.

Consider the family of the following optimal control problems depending on $h \in H$:

$$\begin{aligned} (\mathrm{O})_h \quad & \text{Find } (x_h, u_h) \in X^2 \text{ such that} \\ & F(x_h, u_h, h) = \min \left\{ F(x, u, h) := \textstyle\int_0^1 \varphi(x(t), u(t), h(t))\mathrm{d}t \right\} \\ & \text{subject to} \\ & \dot{x}(t) - f(x(t), u(t), h(t)) = 0 \quad \text{for a.a. } t \in [0, 1], \\ & x(0) = 0, \\ & \vartheta(x(t), h(t)) \le 0 \qquad \text{for all } t \in [0, 1], \end{aligned}$$

where $\varphi\colon\ I\!R^n \times I\!R^m \times I\!R \to I\!R, \quad f\colon\ I\!R^n \times I\!R^m \times I\!R \to I\!R^n, \quad \vartheta\colon\ I\!R^n \times I\!R \to I\!R.$ We assume that the functions $\varphi(\cdot, \cdot, \cdot)$ and $f(\cdot, \cdot, \cdot)$ are as well as $D_x\varphi(\cdot, \cdot, \cdot)$, $D_u\varphi(\cdot, \cdot, \cdot)$, $D_x f(\cdot, \cdot, \cdot)$ and $D_u f(\cdot, \cdot, \cdot)$, are Fréchet differentiable in all arguments. The functions $\vartheta(\cdot, \cdot)$ and $D_x\vartheta(\cdot, \cdot)$ are twice Fréchet differentiable in (x, h).

Moreover, it is assumed, that for a given reference value $\widehat{h} \in H$ of the parameter there exists a reference solution $(\widehat{x}, \widehat{u})$ of $(\mathrm{O})_{\widehat{h}}$, where $\widehat{u} \in C(0, 1; I\!R^m)$. To simplify notation, the functions evaluated at the reference point will be denoted by "hat", e.g., $\widehat{\varphi} := \varphi(\widehat{x}, \widehat{u}, \widehat{h})$, $\widehat{\vartheta} := \vartheta(\widehat{x}, \widehat{h})$.

Remark 1 To minimize technicalities, we consider the fixed homogeneous initial condition. However, the same approach can be applied to general two-points boundary value problems. Also vector-valued state constraints can be considered.

Let us define the following space of multipliers $Y^2 = L^2(0, 1; I\!R^n) \times W^{1,2}(0, 1; I\!R)$ and introduce the following Lagrangian $\mathcal{L}\colon X^2 \times Y^2 \times H \to I\!R$ for $(\mathrm{O})_h$:

$$\mathcal{L}(x, u, p, \mu; h) = F(x, u, h) - (p, \dot{x} - f(x, u, h)) + \mu(0)\vartheta(x(0), h(0))$$
$$+(\dot{\mu}, D_x\vartheta(x, h) f(x, u, h) + D_h\vartheta(x, h)\dot{h}). \quad (1)$$

Remark 2 The Lagrangian is in the so called *indirect* or Pontryagin form, with the absolutely continuous adjoint variable (see Sect. 5 in [4], as well as [3,12]). The state constraints are considered in the space $W^{1,2}(0, 1; I\!R)$, where the general form of a linear functional is given by the inner product. Hence, using the state equation, we get

$$\mu(0)\vartheta(x(0), h(0)) + (\dot{\mu}, \frac{\mathrm{d}}{\mathrm{d}t}\vartheta(x, h))$$
$$= \mu(0)\vartheta(x(0), h(0)) + (\dot{\mu}, D_x\vartheta(x, h) f(x, u, h) + D_h\vartheta(x, h)\dot{h}).$$

Denote by $K = \{d \in W^{1,2}(0, 1; I\!R) \mid d(t) \le 0\}$ the cone of nonpositive functions in $W^{1,2}(0, 1; I\!R)$. The cone polar to K is given (see, e.g., [13]) by :

$$K^+ = \{W^{1,2}(0, 1; I\!R) \mid \mu(0) - \dot{\mu}(0) \ge 0,\ \dot{\mu}(t) \ge 0 \text{ and } \dot{\mu}(\cdot) \text{ is nonincreasing}\}. \quad (2)$$

Clearly, if $\mu \in W^{2.2}(0, 1; I\!R)$, the last condition in (2) reduces to $\ddot{\mu}(t) \le 0$ for almost all $t \in [0, 1]$. Denote by

$$\mathcal{N}_{K^+}(\mu) := \begin{cases} \{y \in W^{1,2}(0, 1; I\!R) \mid (y, \nu - \mu)_{1.2} \le 0\ \forall \nu \in K^+\}, & \text{if } \mu \in K^+, \\ \emptyset, & \text{if } \mu \notin K^+ \end{cases}$$

the normal cone to K^+ at μ.

The stationarity conditions of Lagrangian (1) can be expressed by the following system:

$$\left.\begin{aligned} &\dot{p} + D_x f^*(x, u, h)p + D_x\varphi(x, u, h) + (D_x f^*(x, u, h)D_x\vartheta^*(x, h) \\ &\quad + D^2_{xx}\vartheta(x, h) f(x, u, h) + D^2_{hx}\vartheta(x, h)\dot{h})\dot{\mu} = 0, \quad p(1) = 0, \\ &D_u\varphi(x, u, h) + D_u f^*(x, u, h)p + D_u f^*(x, u, h)D_x\vartheta^*(x, h)\dot{\mu} = 0, \\ &\vartheta(x, h) \in \mathcal{N}_{K^+}(\mu). \end{aligned}\right\} \quad (3)$$

For the sake of simplicity, we will denote $\zeta = (x, u) \in X^2$, $\lambda = (p, \mu) \in Y^2$.

The purpose of this paper is to study the local properties of the map $H \ni h \mapsto (\zeta_h, \lambda_h) \in Z^2$. More precisely: we are looking for conditions, under which there exist a constant $\pi > 0$ and a subset $\mathcal{Z} \subset X^2 \times Y^2$, containing the reference point $(\widehat{\zeta}, \widehat{\lambda})$, such that, for each $h \in \mathcal{B}^H_\pi(\widehat{h})$ there exists a unique stationary point $(\zeta_h, \lambda_h) \in \mathcal{Z}$, which is a Lipschitz continuous function of h.

To cope with this problem, we will need several assumptions to be satisfied at the reference point. These assumptions consist of constraint qualifications and coercivity conditions. To formulate constraint qualifications, for a fixed $\alpha \ge 0$ introduce the sets of indices of α-active constraints $M_\alpha = \{t \in [0, 1] \mid \vartheta(\widehat{x}(t), \widehat{h}(t)) \ge -\alpha\}$. Assume:

(A1) There exists $\alpha > 0$ such that $0 \notin M_\alpha$.
(A2) (*Linear independence*) There exist $\alpha > 0$ and $\chi > 0$ such that

$$|D_u \widehat{f}^*(t) D_x \widehat{\vartheta}^*(t)| \ge \chi \quad \text{for all } t \in M_\alpha.$$

Note that by (A2) we restrict ourselves to the so called *first-order state constraints* [4]. By Theorem 4.3 in [7] we have:

Lemma 1 *If assumptions* (A1) *and* (A2) *are satisfied, then there exists a unique Lagrange multiplier* $\widehat{\lambda} = (\widehat{p}, \widehat{\mu}) \in Y^2$ *such that the first-order optimality conditions* (3) *hold at* $(\widehat{x}, \widehat{u}, \widehat{p}, \widehat{\mu})$.

In addition to the constraint qualifications, we will need some coercivity conditions. Assume:

(A3) (*Legandre–Clebsch condition*) There exists $\bar{\gamma} > 0$ such that

$$\langle v, D^2_{uu}\widehat{\mathcal{L}}(t)\, v\rangle \geq \bar{\gamma}|v|^2 \quad \text{for all } v \in I\!R^m \text{ and all } t \in [0, 1].$$

The following regularity result follows from Theorem 2.1 in [3] (see Proposition 6.6 in [5]):

Lemma 2 *If assumptions* (A1)–(A3) *are satisfied, then* $\widehat{u}, \dot{\widehat{x}}, \dot{\widehat{p}}, \dot{\widehat{\mu}}$ *are Lipschitz continuous on* [0, 1]*, with the Lipschitz modulus denoted by* $\varsigma > 0$.

Denote $\Xi := \left\{(x, u, p, \mu) \in X^2 \times Y^2 \mid \|\ddot{x}\|_\infty, \|\dot{u}\|_\infty, \|\ddot{p}\|_\infty, \|\ddot{\mu}\|_\infty \leq \varsigma\right\}$.

In view of the uniqueness and regularity of $\widehat{\mu}$, we can introduce the following open sets, depending on the parameter $\alpha > 0$:

$$N_\alpha = [0, 1] \setminus \overline{\{t \in [0, 1] \mid -\ddot{\widehat{\mu}}(t) \leq \alpha\}}, \quad \text{as well as } N_0 = \bigcup_{\alpha > 0} N_\alpha.$$

Define

$$\mathcal{E}_\alpha = (y, v) \in X^2 \mid \left\{\begin{array}{l} \dot{y}(t) - D_x\widehat{f}(t)y(t) - D_u\widehat{f}(t)v(t) = 0, \\ \langle D_x\widehat{\vartheta}(t), y(t)\rangle = 0 \;\forall t \in N_\alpha, \\ \langle D_x\widehat{\vartheta}(1), y(1)\rangle = 0 \quad \text{if } \dot{\widehat{\mu}}(1) > 0. \end{array}\right\} \tag{4}$$

For the sake of simplicity we will denote $D^2\widehat{\mathcal{L}} := D^2_{(x,u)(x,u)}\mathcal{L}(\widehat{x}, \widehat{u}, \widehat{p}, \widehat{\mu}; \widehat{h})$.
We assume:

(A4) (*Coercivity*) There exist constants $\alpha > 0$ and $\gamma > 0$ such that

$$\big((y, v), D^2\widehat{\mathcal{L}}(y, v)\big) \geq \gamma(\|y\|^2_{1,2} + \|v\|^2_2) \quad \text{for all } (y, v) \in \mathcal{E}_\alpha. \tag{5}$$

Remark 3 The coercivity condition (A4) is weakened by taking into account strongly active state constraints. It is weaker than the strong coercivity condition, where the active inequality constraints are ignored. The latter condition was used in stability analysis in [1,5]. The application of the weaker condition (A4) is the main new contribution of this paper.

The following result is proved in [10].

Lemma 3 *Suppose that* (A1)–(A3)*, as well as* (A4) *with* $\alpha = 0$*, hold. Then* $(\widehat{x}, \widehat{u})$ *is a second-order local minimizer of* $(\mathrm{O})_{\widehat{h}}$.

3 Stability results

The main tool in stability analysis for constrained processes is Robinson's implicit function theorem for strongly regular generalized equations [14]. The theorem allows to deduce local stability of the stationary points of nonlinear optimization problems from such a stability for the linear-quadratic accessory problems. In [1,5], a modification of Robinson's theorem was used. In this modification, the difficulties connected with the so called *two-norm discrepancy* (see [11]) are overcome by taking into account the additional information on the regularity of the stationary points. In [10], the same approach was used, however, the stability analysis for the accessory problems was different. The weakened second-order condition (A4) was applied there. We will present main steps of this stability analysis for the accessory problem, refering to [1,5,10] for the application of the implicit function theorem.

Let us introduce the following perturbed accessory problem $(\mathrm{AO})_\delta$ for $(\mathrm{O})_{\widehat{h}}$:

$$
\begin{aligned}
(\mathrm{AO})_\delta \;\; & \text{Find } \eta_\delta := (y_\delta, v_\delta) \in X^2 \text{ such that} \\
& J(y_\delta, v_\delta; \delta) = \min J(y, v; \delta) \quad \text{subject to} \\
& \dot{y}(t) - D_x \widehat{f}(t) y(t) - D_u \widehat{f}(t) v(t) - (\widehat{\delta}_3(t) + \delta_3(t)) = 0, \\
& D_x \widehat{\vartheta}(t) y(t) - (\widehat{\delta}_4(t) + \delta_4(t)) \le 0 \quad \text{for all } t \in [0,1],
\end{aligned}
$$

where $J(y, v; \delta) = \frac{1}{2}((y, v), D^2\widehat{\mathcal{L}}(y, v)) - (\widehat{\delta}_1 + \delta_1, y) - (\widehat{\delta}_2 + \delta_2, v)$,
$\widehat{\delta} = (\widehat{\delta}_1, \widehat{\delta}_2, \widehat{\delta}_3, \widehat{\delta}_4)$ is a given function depending on the reference solution, whereas

$$\delta = (\delta_1, \delta_2, \delta_3, \delta_4) \in \Delta := L^2(0,1; I\!R^n) \times L^2(0,1; I\!R^m) \times L^2(0,1; I\!R^n) \times W^{1,2}(0,1; I\!R)$$

is the functional perturbation.

To analyze stability of the stationary points of $(\mathrm{AO})_\delta$, with respect to δ, an important lemma on stability of the coercivity condition (A4) is used. To formulate that lemma, let us choose an arbitrary open set $D \in [0, 1]$ and introduce the following superset of the set $\mathcal{E}_\alpha$ defined in (4):

$$
\mathcal{E}_{\alpha,D} = \{(y, v) \in X^2 \mid \left\{
\begin{array}{l}
\dot{y}(t) - D_x \widehat{f}(t) y(t) - D_u \widehat{f}(t) v(t) = 0, \\
\langle D_x \widehat{\vartheta}(t), y(t) \rangle = 0 \quad \forall t \in N_\alpha \setminus D, \\
\langle D_x \widehat{\vartheta}(1), y(1) \rangle = 0 \quad \text{if } \dot{\widehat{\mu}}(1) > 0.
\end{array}
\right\} \tag{6}
$$

Lemma 4 *Suppose that* (A1)–(A4) *hold. There exist constant* $\beta > 0$ *and* $\kappa > 0$ *such that, if*

$$|t'' - t'| \le \beta \quad \textit{for all } (t', t'') \subset N_\alpha \cap D \tag{7}$$

then

$$\big((y, v), D^2\widehat{\mathcal{L}}(y, v)\big) \ge \kappa(\|y\|_{1,2}^2 + \|v\|_2^2) \quad \textit{for all } (y, v) \in \mathcal{E}_{\alpha,D}. \tag{8}$$

Lemma 4 shows that the coercivity property (A4) of the Hessian of Lagrangian is retained on a broader set $\mathcal{E}_{\alpha,D}$, provided that condition (7) is satisfied.

For any $\nu \in W^{1,2}(0, 1; I\!R)$ denote

$$D_\nu = \{t \in [0,1] \mid \dot{\nu}(\cdot) \text{ is constant a.e. in a neighborhood of } t\}.$$

Lemma 5 *Choose any* $\eta > 0$ *and* $\nu \in K^+$ *such that* $\|\nu - \widehat{\mu}\|_{1,2} \le \eta$. *Let* $(t', t'') \subset D_\nu$ *be any subinterval contained in* D_ν. *Then*

$$\operatorname{meas}\big([t', t''] \cap N_\alpha\big) \le \left(12\frac{\eta^2}{\alpha^2}\right)^{1/3}.$$

Lemmas 4 and 5 imply

Proposition 1 *Let assumptions* (A1)–(A4) *be satisfied. If, for a certain* $\delta \in \Delta$, *there exists a stationary point* $(y_\delta, v_\delta, q_\delta, \nu_\delta)$ *of* $(\mathrm{AO})_\delta$, *which belongs to* $\mathcal{B}_\eta^{X^2 \times Y^2}(\widehat{x}, \widehat{u}, \widehat{p}, \widehat{\mu})$, *where* $\eta = \frac{\alpha}{2}\sqrt{\frac{\beta^3}{3}}$, *then* (y_δ, v_δ) *is a solution of* $(\mathrm{AO})_\delta$.

Using Lemmas 4 and 5, the following stability result for $(AO)_\delta$ is proved in [10]:

Proposition 2 *If assumptions* (A1)–(A4) *are satisfied, then there exist constants* $\theta > 0$, $\tau > 0$ *and* $l > 0$, *such that, for each* $\delta \in \mathcal{B}^{\Delta}_{\theta}(0)$, *there is a unique stationary point* $(y_\delta, v_\delta, q_\delta, \nu_\delta)$ *of* $(AO)_\delta$, *which belongs to* $\mathcal{B}^{X^2\times Y^2}_{\tau}(\widehat{x}, \widehat{u}, \widehat{p}, \widehat{\mu})$, *and*

$$\begin{aligned}&\|y_{\delta'} - y_{\delta''}\|_{1,2}, \|v_{\delta'} - v_{\delta''}\|_2, \|q_{\delta'} - q_{\delta''}\|_{1,2}, \|\nu_{\delta'} - \nu_{\delta''}\|_{1,2}\\ &\le l\|\delta' - \delta''\|_\Delta \quad \text{for all } \delta', \delta'' \in \mathcal{B}^{\Delta}_{\theta}(0).\end{aligned} \tag{9}$$

Outline of the proof

(1) Denote by $\mathcal{D}_\beta$ the family of all open subsets D of $(0, 1)$ such that (7) holds. For a fixed $D \in \mathcal{D}_\beta$, we introduce the following modification of problem $(AO)_\delta$:

$$\begin{aligned}(AO)^D_\delta \quad &\text{Find } \eta^D_\delta := (y^D_\delta, v^D_\delta) \in X^2 \text{ such that}\\ &J(y^D_\delta, v^D_\delta; \delta) = \min J(y, v; \delta) \quad \text{subject to}\\ &\dot y(t) - D_x\widehat{f}(t)y(t) - D_u\widehat{f}(t)v(t) - (\widehat{\delta}_3(t) + \delta_3(t)) = 0,\\ &D_x\widehat{\vartheta}(t)y(t) - (\widehat{\delta}_4(t) + \delta_4(t)) \begin{cases} = 0 & \text{for } t \in N_\alpha\backslash D,\\ \le 0 & \text{for } t \in [0,1]\backslash(N_\alpha\backslash D),\end{cases}\\ &D_x\widehat{\vartheta}(1)y(1) - (\widehat{\delta}_4(1) + \delta_4(1)) = 0, \quad \text{if } \dot{\widehat{\mu}}(1)>0.\end{aligned}$$

Note that $(AO)^D_\delta$ differs from $(AO)_\delta$ only in the form of the inequality constraints. In view of Lemma 4, the quadratic term in the cost functional $J(y, v, ; \delta)$ of $(AO)^D_\delta$ satisfies the coercivity condition (8) on the linear hull of the constraints. Stability of coercive linear-quadratic problems was studied, among others, in [1]. By Lemmas 3.8 and 3.10 in [1], the coercivity condition (8), together with the constraint qualifications (A1) and (A2), ensure that there exists $\rho > 0$, such that, for each $\delta \in \mathcal{B}^{\Delta}_{\rho}(0)$ there is a unique stationary point $(x^D_\delta, u^D_\delta, p^D_\delta, \mu^D_\delta)$ of $(AO)^D_\delta$, where (x^D_δ, u^D_δ) is the solution. Moreover, the stationary points are Lipschitz continuous in δ, uniformly with respect to $D \in \mathcal{D}_\beta$. Since $(\widehat{x}, \widehat{u}, \widehat{p}, \widehat{\mu})$ is a stationary point of $(AO)^D_0$, we find that there exist constants $r > 0$ and $\rho > 0$, independent of D, such that

$$\begin{aligned}&\|x^D_\delta - \widehat{x}\|_{1,2}, \|u^D_\delta - \widehat{u}\|_2, \|p^D_\delta - \widehat{p}\|_{1,2}, \|\mu^D_\delta - \widehat{\mu}\|_{1,2}\\ &\le r\|\delta\|_\Delta \quad \text{for all } \delta \in \mathcal{B}^{\Delta}_{\rho}(0).\end{aligned} \tag{10}$$

(2) Set

$$\theta \in \left(0, \min\left\{\rho, \frac{\alpha}{r}\sqrt{\frac{\beta^3}{48}}\right\}\right), \tag{11}$$

where $\beta > 0$ is given in (7). Choose an arbitrary $\delta \in \mathcal{B}^{\Delta}_{\theta}(0)$ and an arbitrary $D \in \mathcal{D}_\beta$. Denote $P^D_\delta := \{t \in [0,1] \mid \dot\mu^D_\delta(\cdot) = \text{const in a neighborhood of } t\}$. By Lemma 5 and by (10)–(11), there exists $\beta' \in (0, \beta/2)$ such that

$$\text{meas}\,([t', t''] \cap N_\alpha) \le \beta' < \beta/2 \quad \text{for any } [t', t''] \subset P^D_\delta. \tag{12}$$

Thus, condition (7) is satisfied with a margin.
Let $\{(x^D_\delta, u^D_\delta)\}$ be a sequence of solutions to $(O)^D_\delta$ minimizing $J(x^D_\delta, u^D_\delta; \delta)$ with respect to $D \in \mathcal{D}_\beta$, i.e.,

$$\lim_D J(x^D_\delta, u^D_\delta; \delta) = \inf_{D\in\mathcal{D}_\beta} J(x^D_\delta, u^D_\delta; \delta) := \bar J_\delta. \tag{13}$$

It follows from (8) and (10) that $\bar{J}_\delta$ is finite and the set $\{(x_\delta^D, u_\delta^D) \in X^2 \mid D \in \mathcal{D}_\beta\}$ is weakly compact in X^2. Hence we can extract a weakly convergent subsequence, still denoted by $\{(x_\delta^D, u_\delta^D)\}$. Thus, there exists a pair $(y_\delta, v_\delta) \in X^2$ such that:

$$\left.\begin{array}{l} u_\delta^D \rightharpoonup v_\delta \text{ weakly in } L^2(0,1;I\!R^m), \\ x_\delta^D \rightharpoonup y_\delta \text{ weakly in } W_0^{1,2}(0,1;I\!R^n), \text{ i.e., strongly in } C(0,1;I\!R^n). \end{array}\right\} \tag{14}$$

It follows from (14) that (y_δ, v_δ) is feasible for $(\mathrm{AO})_\delta^{D_\delta}$, where

$$D_\delta := \{t \in [0,1] \mid D_x\widehat{\vartheta}(t)y_\delta(t) - (\widehat{\delta}_4(t) + \delta_4(t)) < 0\}.$$

Using (8) and (13) we show that (y_δ, v_δ) is the solution of $(\mathrm{AO})_\delta^{D_\delta}$.

(3) Denote by $(q_\delta, \nu_\delta) \in Y^2$ the unique Lagrange multiplier of $(\mathrm{AO})_\delta^{D_\delta}$ associated with (y_δ, v_δ). Using (12) and (13), we show that $\nu_\delta \in K^+$, where K^+ is defined in (2). In turn, it implies that $(y_\delta, v_\delta, q_\delta, \nu_\delta)$ is a stationary point of $(\mathrm{AO})_\delta$.

(4) Using (8), we show that (9) holds, which, in particular implies the uniqueness of the stationary points $(y_\delta, v_\delta, q_\delta, \nu_\delta)$.

Proposition 2, together with the regularity results for the stationary points of $(\mathrm{AO})_\delta$, obtained in [10] (see Lemmas 4.1 and 4.5 therein), show that the implicit function theorem can be applied to $(\mathrm{O})_h$, and by that theorem we get:

Theorem 1 *If assumptions* (A1)–(A4) *are satisfied, then there exist constants* $\pi > 0$, $\tau > 0$ *and* $\ell > 0$, *such that, for each* $h \in \mathcal{B}_\pi^H(\widehat{h})$, *there is a unique stationary point* $(\zeta_h, \lambda_h) := (x_h, u_h, p_h, \mu_h) \in \Xi \cap \mathcal{B}_\tau^{X^2\times Y^2}(\widehat{\zeta}, \widehat{\lambda})$ *of* $(\mathrm{O})_h$ *and*

$$\begin{array}{c} \|x_{h'} - x_{h''}\|_{1,2}, \|u_{h'} - u_{h''}\|_2, \|p_{h'} - p_{h''}\|_{1,2}, \|\mu_{h'} - \mu_{h''}\|_{1,2} \le \ell\|h' - h''\|_Z \\ \text{for all } h', h'' \in \mathcal{B}_\pi^H(\widehat{h}). \end{array} \tag{15}$$

Note that in Theorem 1 two norms in the space of parameters are involved. We consider h belonging to a ball in the space H, whereas in the Lipschitz estimates (15) there appears the norm of the space Z.

We still have to prove that (x_h, u_h) is a solution of $(\mathrm{O})_h$. To this end we use Lemma 4 and Theorem 1 to show that the second-order sufficient optimality condition (5) holds, for all h in a neighborhood of $\widehat{h}$. Thus, we get:

Corollary 1 *If assumptions* (A1)–(A4) *are satisfied, then, for* $\pi > 0$ *sufficiently small, the stationary point* (ζ_h, λ_h) *in Theorem* 1 *corresponds to a solution and Lagrange multiplier of* $(\mathrm{O})_h$.

Acknowledgements Supported by the Polish Ministry of Scientific Research and Information Technology grant 3 T11C 051 28.

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